

# Vertex-transitive Haar graphs that are not Cayley graphs

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*Dedicated to Egon Schulte and Károly Bezdek on the occasion of their 60th birthdays*

## Abstract

In a recent paper (arXiv:1505.01475 ) Estélyi and Pisanski raised a question whether there exist vertex-transitive Haar graphs that are not Cayley graphs. In this note we construct an infinite family of trivalent Haar graphs that are vertex-transitive but non-Cayley. The smallest example has 40 vertices and is the well-known Kronecker cover over the dodecahedron graph  $G(10, 2)$ , occurring as the graph ‘40’ in the Foster census of connected symmetric trivalent graphs.

*Keywords:* Haar graph, Cayley graph, vertex-transitive graph.

*MSC 2010:* 05E18 (primary), 20B25 (secondary).

## 1 Introduction

Let  $G$  be a group, and  $S$  be a subset of  $G$  with  $1_G \notin S$ . Then the *Cayley graph*  $\text{Cay}(G, S)$  is the graph with vertex-set  $G$  and with edges of the form  $\{g, sg\}$  for all  $g \in G$  and  $s \in S$ . Equivalently, since all edges can be written in the form  $\{1, s\}g$ , this is a covering graph over a single-vertex graph having loops and semi-edges, with voltages taken from  $S$ : the order of a voltage over a semi-edge is 2 (corresponding to an involution in  $S$ ), while the order of voltage over a loop is greater than 2 (corresponding to a non-involution in  $S$ ). Note that we may assume  $S = S^{-1}$ .

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A natural generalisation of Cayley graphs are the so called *Haar graphs*, introduced in [14] by Hladnik et al, as follows. A *dipole* is a graph with two vertices, say black and white, and parallel edges (each from the *white* vertex to the *black* vertex), but no loops. Given a group  $G$  and an arbitrary subset  $S$  of  $G$ , the *Haar graph*  $H(G, S)$  is the regular  $G$ -cover of a dipole with  $|S|$  parallel edges, labeled by elements of  $S$ . In other words, the vertex-set of  $H(G, S)$  is  $G \times \{0, 1\}$ , and the edges are of the form  $\{(g, 0), (sg, 1)\}$  for all  $g \in G$  and  $s \in S$ . If it is not ambiguous, we use the notation  $(x, 0) \sim (y, 1)$  to indicate an edge  $\{(x, 0), (y, 1)\}$  of  $H(G, S)$ . The name ‘Haar graph’ comes from the fact that when  $G$  is an abelian group, the Schur norm of the corresponding adjacency matrix can be easily evaluated via the so-called Haar integral on  $G$  (see [13]).

Note that the group  $G$  acts on  $H(G, S)$  as a group of automorphisms, by right multiplication, and moreover,  $G$  acts regularly on each of the two parts of  $H(G, S)$ , namely  $\{(g, 0) : g \in G\}$  and  $\{(g, 1) : g \in G\}$ . Conversely, if  $\Gamma$  is any bipartite graph and its automorphism group  $\text{Aut } \Gamma$  has a subgroup  $G$  that acts regularly on each part of  $\Gamma$ , then  $\Gamma$  is a Haar graph — indeed  $\Gamma$  is isomorphic to  $H(G, S)$  where  $S$  is determined by the edges incident with a given vertex of  $\Gamma$ .

Haar graphs form a special subclass of the more general class of *bi-Cayley graphs*, which are graphs that admit a semiregular group of automorphisms with two orbits of equal size. Every bi-Cayley graph can be realised as follows. Let  $L$  and  $R$  be subsets of a group  $G$  such that  $L = L^{-1}$ ,  $R = R^{-1}$  and  $1 \notin L \cup R$ , and let  $S$  be any subset of  $G$ . Now take a dipole with edges labelled by elements of  $S$ , and add  $|L|$  loops to the white (or ‘left’) vertex and label these by elements of  $L$ , and similarly add  $|R|$  loops to the black (or ‘right’) vertex and label these by elements of  $R$ . This is a voltage graph, and the bi-Cayley graph  $\text{BCay}(G, L, R, S)$  is its regular  $G$ -cover. The vertex-set of  $\text{BCay}(G, L, R, S)$  is  $G \times \{0, 1\}$ , and the edges are of three forms:  $\{(g, 0), (lg, 0)\}$  for  $l \in L$ ,  $\{(g, 1), (rg, 1)\}$  for  $r \in R$ , and  $\{(g, 0), (sg, 1)\}$  for  $s \in S$ , for all  $g \in G$ . Note that the Haar graph  $H(G, S)$  is exactly the same as the bi-Cayley graph  $\text{BCay}(G, \emptyset, \emptyset, S)$ .

Recently bi-Cayley graphs (and Haar graphs in particular) have been investigated by several authors — see [8, 9, 10, 14, 15, 16, 17, 18, 19, 21, 22, 23, 26, 28], for example.

It is known that every Haar graph over an abelian group is a Cayley graph (see [21]). More precisely, if  $A$  is an abelian group, then a Haar graph over  $A$  is a Cayley graph over the corresponding *generalised dihedral group*  $D(A)$ , which is the group generated by  $A$  and the automorphism of  $A$  that inverts every element of  $A$  (see [24]). The authors of [14] considered only cyclic Haar graphs — that is, Haar graphs  $H(G, S)$  where  $G$  is a cyclic group. In [9], the second and third authors of this paper extended the study of Haar graphs to those over non-abelian groups, and found some that are not vertex-transitive, and some others that are Cayley graphs. The existence of Haar graphs that are vertex-transitive but non-Cayley remained open, and led to the following question.

**Problem 1.** *Is there a non-abelian group  $G$  and a subset  $S$  of  $G$  such that the Haar graph  $H(G, S)$  is vertex-transitive but non-Cayley?*

In this note we give a positive answer to the above question, by exhibiting an infinite family of trivalent examples, coming from a family of double covers of generalised Petersen graphs. These graphs, which we denote by  $D(n, r)$  for any integers  $n$  and  $r$  with

$n \geq 3$  and  $0 < r < n$ , are described in Section 2. They have been considered previously by other authors (as we explain); in particular, by a theorem of Feng and Zhou [28], it is known exactly which of the graphs  $D(n, r)$  are vertex-transitive, and which are Cayley. Then in Section 3 we determine necessary and sufficient conditions for  $D(n, r)$  to be a Haar graph, and this provides the answer in Section 4.

## 2 The graphs $D(n, r)$ and their properties

Let  $G(n, r)$  be the generalised Petersen graph on  $2n$  vertices with span  $r$ . By  $D(n, r)$  we denote a double cover of  $G(n, r)$ , in which the edges get non-trivial voltage if and only if they belong to the ‘inner rim’ (see below). This gives a class of graphs that was introduced by Zhou and Feng [27] under the name of *double generalised Petersen graphs*, and studied recently also by Kutnar and Petecki [20]. In both [27] and [20], the notation  $DP(n, r)$  was used for the graph  $D(n, r)$ . But it is easy for us to define the vertices and edges of the graph  $D(n, r)$  explicitly.

There are four types of vertices, called  $u_i, v_i, w_i$  and  $z_i$  (for  $i \in \mathbb{Z}_n$ ), and three types of edges, given by the sets

$$\begin{aligned} \Omega &= \{ \{u_i, u_{i+1}\}, \{z_i, z_{i+1}\} : i \in \mathbb{Z}_n \} && \text{(the ‘outer’ edges),} \\ \Sigma &= \{ \{u_i, v_i\}, \{w_i, z_i\} : i \in \mathbb{Z}_n \} && \text{(the ‘spokes’), and} \\ I &= \{ \{v_i, w_{i+r}\}, \{v_i, w_{i-r}\} : i \in \mathbb{Z}_n \} && \text{(the ‘inner’ edges).} \end{aligned}$$

This specification makes it easy to see that each  $D(n, r)$  is a special tetracirculant [11], which is a cyclic cover  $\Sigma_0(n, a, k, b)$  over the voltage graph given in Figure 1. To see this, simply take  $a = b = 1$  and  $k = 2r$ , and then  $D(n, r) \cong \Sigma_0(n, 1, 2r, 1)$ .

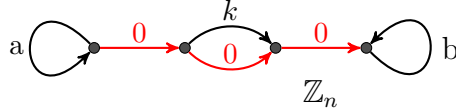


Figure 1: Voltage graph defining the tetracirculant  $\Sigma_0(n, a, k, b)$

We now give some of other properties of the graphs  $D(n, r)$ .

**Proposition 1.** *Every  $D(n, r)$  is connected.*

*Proof.* Clearly all of the  $u_i$  lie in the same component as each other, as do all the  $z_j$ . Next, all the  $v_i$  lie in the same component as the  $u_i$ , and similarly, all the  $w_j$  lie in the same component as the  $z_j$ . Finally, there are edges between the vertices  $v_i$  and some of the  $w_j$ , and this makes the whole graph connected.  $\square$

**Proposition 2.** *The graph  $D(n, r)$  is bipartite if and only if  $n$  is even.*

*Proof.* If  $n$  is odd, then the vertices  $u_i$  lie in a cycle of odd length, and so the graph is not bipartite. On the other hand, if  $n$  is even, then the graph is bipartite, with one part containing the vertices  $u_i$  and  $w_{i \pm r}$  for even  $i$  and the vertices  $v_j$  and  $z_{j \pm r}$  for odd  $j$ .  $\square$

We now consider automorphisms of the graphs  $D(n, r)$ . Some automorphisms are apparent from the definition, such as these, which were noted in [20]:

$$\begin{aligned}\alpha : \quad & u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1}, \quad w_i \mapsto w_{i+1}, \quad z_i \mapsto z_{i+1} && (\text{rotation}), \\ \beta : \quad & u_i \mapsto z_i, \quad v_i \mapsto w_i, \quad w_i \mapsto v_i, \quad z_i \mapsto u_i && (\text{flip symmetry}), \\ \gamma : \quad & u_i \mapsto u_{-i}, \quad v_i \mapsto v_{-i}, \quad w_i \mapsto w_{-i}, \quad z_i \mapsto z_{-i} && (\text{reflection}).\end{aligned}$$

Immediately we obtain the following:

**Proposition 3.** *The automorphism group of the graph  $D(n, r)$  has at most two orbits on vertices, namely the set of all  $u_i$  and all  $z_j$ , and the set of all  $v_i$  and all  $w_j$ .*

Note also that  $\alpha$  and  $\beta$  commute with each other. In fact, Zhou and Feng [27] proved that  $D(n, r)$  is isomorphic to the bi-Cayley graph  $\text{BCay}(G, R, L, \{1\})$  over the abelian group  $G = \langle \alpha, \beta \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ , and  $R = \{\alpha, \alpha^{-1}\}$  and  $L = \{\alpha^r \beta, \alpha^{-r} \beta\}$ .

Next, we consider isomorphisms among the graphs  $G(n, r)$  and  $D(n, r)$ .

**Proposition 4.** *For every  $n$  and  $r$ , the graph  $D(n, r)$  is isomorphic to  $D(n, n-r)$ , and  $D(2n, r)$  is isomorphic to  $D(2n, n-r)$ .*

*Proof.* First, the graphs  $D(n, r) \cong D(n, n-r)$  because  $G(n, r)$  is identical to  $G(n, n-r)$ . For the second part, consider a 180 degree rotation of the two ‘inner’ layers, namely  $w_i \mapsto w_{i+n}$  and  $z_i \mapsto z_{i+n}$  for all  $i$ . This shows that  $D(2n, r)$  is isomorphic to  $D(2n, n+r)$ , and then applying the first part gives  $D(2n, r) \cong D(2n, 2n - (n+r)) = D(2n, n-r)$ .  $\square$

Here we note that it can happen that the graphs  $D(n, r)$  and  $D(n, s)$  are different when  $G(n, r)$  is isomorphic to  $G(n, s)$ . For instance,  $G(7, 2)$  is isomorphic to  $G(7, 3)$  but  $D(7, 2)$  is not isomorphic to  $D(7, 3)$ , since  $D(7, 3)$  is planar but  $D(7, 2)$  is not.

Also we have the following:

**Proposition 5.** *For every  $r$ , the graph  $D(2r+1, r)$  is planar, and isomorphic to the generalised Petersen graph  $G(4r+2, 2)$ .*

*Proof.* To see that  $D(2r+1, r)$  is planar, first note that since  $r$  is coprime to  $2r+1$ , the edges between the vertices  $v_i$  and  $w_j$  give a cycle of length  $2(2r+1)$ , namely  $(v_0, w_{-r}, v_1, w_{1-r}, v_2, w_{2-r}, \dots, v_{-2}, w_{r-1}, v_{-1}, w_r)$ . Now draw three concentric circles, with the middle one for this  $2(2r+1)$ -cycle, the inside one for the  $(2r+1)$ -cycle  $(u_0, u_1, \dots, u_{2r})$ , and the outside one for the  $(2r+1)$ -cycle  $(z_0, z_1, \dots, z_{2r})$ , in a consistent order, and then add the spoke edges  $\{u_i, v_i\}$  and  $\{w_i, z_i\}$  in the natural way. In the resulting planar drawing of  $D(2r+1, r)$ , there is an inner face of length  $2r+1$  (with the  $u_i$  as vertices), then two layers of pentagonal faces (bounded by cycles of the form  $(u_i, v_i, w_{i-r}, v_{i+1}, u_{i+1})$  and  $(v_j, w_{j+r}, z_{j+r}, z_{j-r}, w_{j-r})$ ), and an outer face of length  $2r+1$  (with the  $z_j$  as vertices). After doing this, it is also easy to see that  $D(2r+1, r)$  is isomorphic to the generalised Petersen graph  $G(4r+2, 2)$ , with the spoke edges joining vertices of the large  $2(2r+1)$ -cycle (on the vertices  $v_i$  and  $w_j$ ) to the two  $(2r+1)$ -cycles (on the vertices  $u_i$  and vertices  $z_j$  respectively).  $\square$

In particular, the graph  $D(5, 2)$  is isomorphic to the dodecahedral graph  $G(10, 2)$ , and hence  $D(5, 2)$  is vertex-transitive. But as we will see, it is not a Haar graph.

Finally in this section, we consider the questions of which of the graphs  $D(n, r)$  are vertex-transitive, and which are Cayley (or equivalently, for which  $\text{Aut}(D(n, r))$  has a subgroup that acts regularly on vertices). Recall that  $\text{Aut}(D(n, r))$  has at most two orbits on vertices, and just one when  $(n, r) = (5, 2)$ . The complete picture was determined by Feng and Zhou in [28, Theorem 1.3], as follows:

**Theorem 6.** *The graph  $D(n, r)$  is vertex-transitive if and only if  $n = 5$  and  $r = \pm 2$ , or  $n$  is even and  $r^2 \equiv \pm 1 \pmod{\frac{n}{2}}$ . In the first case,  $D(n, r)$  is isomorphic to the dodecahedral graph  $G(10, 2)$ , which is non-Cayley, and in the second case, if  $r^2 \equiv 1 \pmod{\frac{n}{2}}$  then  $D(n, r)$  is a Cayley graph, while if  $r^2 \equiv -1 \pmod{\frac{n}{2}}$  then  $D(n, r)$  is non-Cayley.*

### 3 The graphs $D(n, r)$ as Haar graphs

Recall that a Haar graph is a regular cover of a dipole, and also a bi-Cayley graph. Also we have the following, proved in a different way in [9, Proposition 5]:

**Proposition 7.** *A Cayley graph is a Haar graph if and only if it is bipartite.*

*Proof.* Let  $\Gamma$  be a Cayley graph, say for a group  $K$ . Then  $K$  acts on  $\Gamma$  as a group of automorphisms, and acts regularly on the vertices of  $\Gamma$ . Now if  $\Gamma$  is a Haar graph, then by definition  $\Gamma$  is bipartite. Conversely, suppose  $\Gamma$  is bipartite. Then the subgroup  $G$  of  $K$  preserving each of the two parts of  $\Gamma$  has index 2 in  $K$ , and acts regularly on each part, so  $\Gamma$  is a Haar graph (by the argument given in the Introduction).  $\square$

We can now prove our main theorem:

**Theorem 8.**  *$D(n, r)$  is a Haar graph if and only if it is vertex-transitive and  $n$  is even.*

*Proof.* First, we note that  $D(n, r)$  is bipartite if and only if  $n$  is even, by Proposition 2, and hence we may suppose that  $n$  is even, and then show that under that assumption,  $D(n, r)$  is a Haar graph if and only if it is vertex-transitive.

One direction is easy. Suppose  $\Gamma = D(n, r)$  is a Haar graph, say  $H(G, S)$ . Then by the definition of a Haar graph given in the Introduction, the subgroup  $G_R$  of  $\text{Aut } \Gamma$  induced by  $G$  has two orbits on vertices, namely the two parts of the bipartition of  $\Gamma$ . On the other hand, by Proposition 3, all the vertices  $u_i$  lie in the same orbit of  $\text{Aut } \Gamma$ ; and then since these vertices lie in both parts of  $\Gamma$ , it follows that  $\text{Aut } \Gamma$  has a single orbit on vertices. Thus  $\Gamma$  is vertex-transitive.

For the converse, suppose that  $\Gamma = D(n, r)$  is vertex-transitive, and let  $m = \frac{n}{2}$ . Then by Theorem 6, we know that  $r^2 \equiv \pm 1 \pmod{m}$ . Also by Proposition 4 we may suppose that  $0 < r < m$ , and further, we may suppose that  $r$  is odd, because if  $r$  is even then  $m$  is odd, and then by Proposition 4 we can replace  $r$  by  $m - r$ . We now proceed by considering separately the two cases  $r^2 \equiv \pm 1 \pmod{m}$ .

Case (a): Suppose that  $r^2 \equiv 1 \pmod{m}$ . Then by Theorem 6, we know that  $D(n, r)$  is a Cayley graph, and also since it is bipartite, it follows from Proposition 7 that it is a Haar graph as well.

Case (b): Suppose that  $r^2 \equiv -1 \pmod{m}$ . In this case we construct a group of automorphisms of  $D(n, r)$  that acts regularly on each part of  $D(n, r)$ . To do this, we take the automorphism  $\alpha$  from the previous section, given by

$$\alpha : u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1}, \quad w_i \mapsto w_{i+1}, \quad z_i \mapsto z_{i+1},$$

and then take an additional automorphism  $\delta$ , given by

$$\delta : u_i \mapsto v_{ri+1}, \quad v_i \mapsto u_{ri+1}, \quad w_i \mapsto z_{ri+1}, \quad z_i \mapsto w_{ri+1} \quad \text{if } m \text{ is odd and } i \text{ is even,}$$

$$\delta : u_i \mapsto w_{ri+1}, \quad v_i \mapsto z_{ri+1}, \quad w_i \mapsto u_{ri+1}, \quad z_i \mapsto v_{ri+1} \quad \text{if } m \text{ is odd and } i \text{ is odd,}$$

or

$$\delta : u_i \mapsto v_{ri+1}, \quad v_i \mapsto u_{ri+1}, \quad w_i \mapsto z_{ri+m+1}, \quad z_i \mapsto w_{ri+m+1} \quad \text{if } m \text{ is even and } i \text{ is even,}$$

$$\delta : u_i \mapsto w_{ri+1}, \quad v_i \mapsto z_{ri+1}, \quad w_i \mapsto u_{ri+m+1}, \quad z_i \mapsto v_{ri+m+1} \quad \text{if } m \text{ is even and } i \text{ is odd.}$$

It is a straightforward exercise to verify that  $\delta$  preserves the edge-set  $\Omega \cup \Sigma \cup I$  of  $D(n, r)$ , and also preserves the two parts of  $D(n, r)$ , given in the proof of Proposition 2. To do the former, it is important to note that  $r^2 \equiv 1 \pmod{4}$  (because  $r$  is odd), and hence that  $r^2 \equiv -1 \pmod{n}$  when  $m$  is odd, while  $r^2 \equiv m-1 \pmod{n}$  when  $m$  is even. For example, if  $m$  and  $i$  are even then  $\{v_i, w_{i+r}\}^\delta = \{u_{ri+1}, u_{r(i+r)+m+1}\} = \{u_{ri+1}, u_{ri}\}$ .

It is also easy to see that conjugation by  $\delta$  takes  $\alpha^2$  to  $\alpha^{2r}$ , and so the subgroup  $G$  of  $\text{Aut}(D(n, r))$  generated by  $\alpha^2$  and  $\delta$  is isomorphic to the semi-direct product  $\mathbb{Z}_m \rtimes_r \mathbb{Z}_4$ . In particular,  $G$  has order  $4m = 2n$ . Also  $G$  acts transitively and hence regularly on each of the two parts of  $D(n, r)$ , and therefore  $D(n, r)$  is a Haar graph.  $\square$

## 4 Vertex-transitive Haar graphs that are not Cayley graphs

Combining Theorems 6 and 8, we have the following, in answer to Problem 1:

### Theorem 9.

- (a) If  $n$  is odd, or if  $n$  is even and  $r^2 \not\equiv \pm 1 \pmod{\frac{n}{2}}$ , then  $D(n, r)$  is not a Haar graph, and is vertex-transitive only when  $(n, r) = (5, \pm 2)$ ;
- (b) If  $n$  is even and  $r^2 \equiv 1 \pmod{\frac{n}{2}}$ , then  $D(n, r)$  is a Haar graph and a Cayley graph;
- (c) If  $n$  is even and  $r^2 \equiv -1 \pmod{\frac{n}{2}}$ , then  $D(n, r)$  is a Haar graph and is vertex-transitive but not a Cayley graph.

**Corollary 10.** If  $m > 2$  and  $r^2 \equiv -1 \pmod{m}$ , then  $D(2m, r)$  is a Haar graph that is vertex-transitive but non-Cayley. In particular, there are infinitely many such graphs.

*Proof.* The first part follows immediately from Theorem 9, and the second part follows from a well known fact in number theory, namely that  $-1$  is a square mod  $m$  if and only if  $m$  or  $m/2$  is a product of primes  $p \equiv 1 \pmod{4}$  (see [12, Chapter 6]), or simply by taking  $m = r^2 + 1$  for each integer  $r \geq 2$ .  $\square$

We discovered the first few of these examples during the week of the conference *Geometry and Symmetry*, held in 2015 at Veszprém, Hungary, to celebrate the 60th birthdays of Károly Bezdek and Egon Schulte.

The smallest of our examples is  $D(10, 2)$ , of order 40, occurring when  $m = 5$  and  $r \equiv \pm 2$  or  $\pm 3 \pmod{10}$  (noting that  $m - r = 3$  when  $(m, r) = (5, 2)$ ). This is also the smallest known Haar graph that is vertex-transitive and non-Cayley. It is a Kronecker cover over the dodecahedral graph  $G(10, 2)$ , and is also a double cover over the Desargues graph  $G(10, 3)$ . These graphs are illustrated in Figure 2.

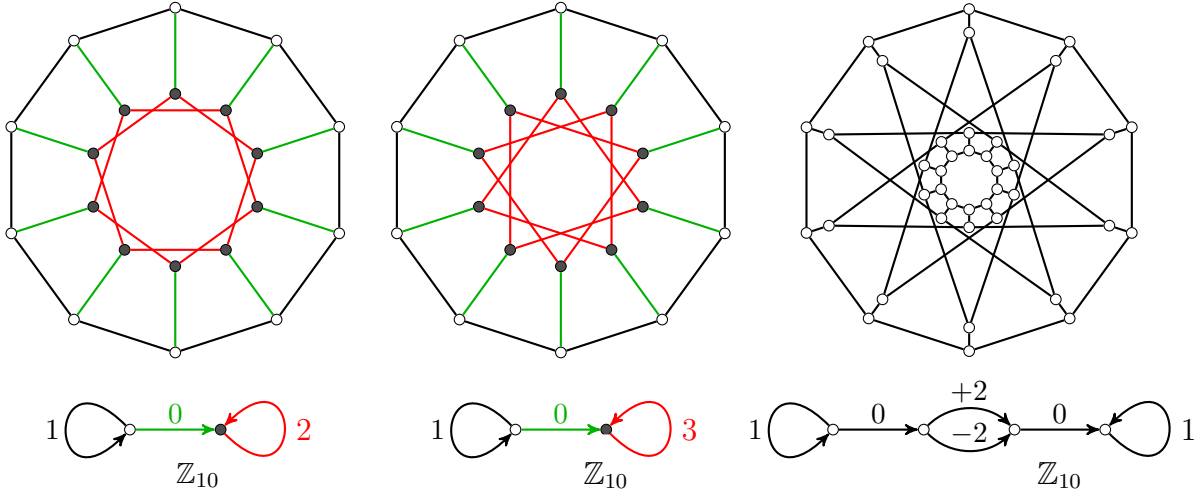


Figure 2: The dodecahedral graph  $G(10, 2)$ , the Desargues graph  $G(10, 3)$ , and the Haar graph  $D(10, 2) \cong D(10, 3) \cong \mathbf{F40}$

The graph  $D(10, 2)$  was known by R.M. Foster as early as the late 1930s, and appears as the graph ‘**40**’ (alternatively known as ‘**F40**’) in the *Foster Census* of connected symmetric trivalent graphs [7]. It was also studied in [25] by Asia Ivić Weiss (the chair of the Veszprém conference), and by Betten, Brinkmann and Pisanski in [1], and Boben, Grünbaum, Pisanski and Žitnik in [2]. It has girth 8, and automorphism group of order 480, and it is not just vertex-transitive, but is also arc-transitive. Moreover, by a very recent theorem of Kutnar and Petecki [20], the graph  $D(n, r)$  is arc-transitive only when  $(n, r) = (5, 2)$  or  $(10, 2)$  or  $(10, 3)$ . This implies that **F40** is the only example from the family of graphs  $D(n, r)$  that is arc-transitive but non-Cayley.

In fact, **F40** is the smallest vertex-transitive non-Cayley Haar graph, in terms of both the graph order and the number of edges. We found this by running a MAGMA [3] computation to construct all Haar graphs with at most 40 vertices or at most 60 edges, with a check for which of the graphs are vertex-transitive but non-Cayley. Incidentally, this computation shows that there are 60 different examples of order 40, with valencies running between 3 and 17, but just one of valency 3, namely **F40**.

Finally, there are many other examples of vertex-transitive non-Cayley Haar graphs that are not of the form  $D(n, r)$ , including 3-valent examples of orders 80, 112, 120 and 128, and higher-valent examples of orders 48, 64, 72, 78 and 80. Among the 3-valent

examples, many are arc-transitive, including the graphs **F80** and **F640** in the Foster census [7] and its extended version in [5, 6], and others in the first author’s complete set of all arc-transitive trivalent graphs of order up to 10000 described on his website [4]. Most of these ‘small’ examples are abelian regular covers of **F40**, of orders 1280, 2560, 3240, 5000, 5120, 6480, 6720, 9720 and 10000, and are 3-arc-regular, but two others are 2-arc-regular of type  $2^2$ , with orders 6174 and 8064, and these are abelian regular covers of the Pappus graph (**F18**) and the Coxeter graph (**F28**) respectively.

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